

A NOTE ON KASPAROV PRODUCTS

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Combining Kasparov's theorem of Voiculescu and Cuntz's description of KK -theory in terms of quasimorphisms, we give a simple construction of the Kasparov product. This will be used in a more general context of locally convex algebras in order to treat products of certain universal cycles.

1 Introduction

The goal of this note is to establish existence of the Kasparov product based on Kasparov's theorem of Voiculescu ([Kas80a]), and to examine how this construction is related to the one used by Kasparov.

In the first section, we interpret the connection condition and the existence of Kasparov product ([Kas80b]) as the existence of a certain extension of a quasimorphism ([Cun87]). Such extensions always exist, as can be seen by applying split exactness of KK to a certain algebra D_α that is a semidirect product of the domain and target of a quasimorphism. The resulting description of the Kasparov product already yields a useful way to construct the Kasparov product; it is particularly well adapted to generalisations of the bimodule-formalism to locally convex algebras, where it may be used to calculate products of certain "smooth" submodules, and is used in [Gre] in a crucial manner.

In the second section, it is shown that, without making use of split exactness of KK , one can, in case that Kasparov's theorem of Voiculescu is available, construct the product by using this interpretation. First we show how to reduce quasimorphisms to a single morphism and a unitary; and if an absorbing morphism is chosen, all classes of quasimorphisms are obtained from it by conjugation by a unitary. Applying this to a pair of composable quasimorphisms, we see that it suffices to extend quasimorphisms to just

one "universal" algebra; if further the domain or target of the first quasihomomorphism is nuclear, there is a canonical way to extend quasihomomorphisms.

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2 The Kasparov product revisited

Quasihomomorphisms were introduced by Cuntz in [Cun83] and further developed in [Cun87].

Definition 1. *Let B be stable, \hat{B} a C^* -algebra containing B as an ideal; then a quasihomomorphism from A to B is a pair of homomorphisms from A to \hat{B} such that $\alpha(a) - \bar{\alpha}(a) \in B$ for all $a \in A$.*

For nonstable B , a quasihomomorphism from A to B is by definition a quasihomomorphism from A into the stabilisation $\mathbb{K} \otimes B$ of B .

Let (E, φ, F) be a Kasparov (A, B) -module with A and B trivially graded. If F is selfadjoint and invertible, then with respect to the grading:

$$\varphi = \begin{pmatrix} \varphi^{(0)} & \\ & \varphi^{(1)} \end{pmatrix} \text{ and } F = \begin{pmatrix} & T^{-1} \\ T & \end{pmatrix}$$

where the $\varphi^{(i)}$ are homomorphisms $A \rightarrow \mathbb{B}_B(E^{(i)})$ and T is by hypothesis a unitary in $\mathbb{B}_B(E^{(0)}, E^{(1)})$. Thence we obtain a quasihomomorphism $(\alpha, \bar{\alpha}) := (\varphi^{(0)}, T^{-1}\varphi^{(1)}T)$ from A to $\mathbb{K}_B(E^{(0)})$ simply by identifying $E^{(0)}$ and $E^{(1)}$ via T , and where we view the latter as a subalgebra of $\mathbb{K} \otimes B$ via the stabilization-theorem.

We may always reduce to this case by using the standard simplifications in KK -theory, and therefore we can define an associated quasihomomorphism $Qh(x)$ to every Kasparov module.

The original construction of the Kasparov product from [Kas80b] was quite technical. We will use the version based on the notion of connection introduced by Connes and Skandalis. We fix the following setting: Let E_1 be a graded Hilbert B -module, E_2 a graded Hilbert C -module, $\varphi : B \rightarrow \mathbb{B}_C(E_2)$ a $*$ -homomorphism and F an odd selfadjoint operator on E_2 . We set $E_{12} := E_1 \otimes_B E_2$, and define for every $x \in E_1$ an operator $T_x : E_2 \rightarrow E_1 \otimes_B E_2$, $y \mapsto x \otimes y$. Note that the adjoint of T_x is given by the mapping $E_{12} \rightarrow E_2$, $y \otimes z \mapsto \varphi(\langle x|y \rangle)z$, and $T_{x'}T_x^* = \theta_{x',x} \otimes \text{id}_{E_2}$.

Definition 2. An E_1 -connection for an odd operator F is an odd selfadjoint operator G such that for all homogeneous $x \in E_1$

$$T_x F - (-1)^{\partial x} G T_x \in \mathbb{K}_C(E_2, E_{12}) \text{ and } F T_x^* - (-1)^{\partial x} T_x^* G \in \mathbb{K}_C(E_{12}, E_2).$$

As a consequence of the stabilisation theorem, such connections exist in case we deal with Kasparov modules; more precisely:

Proposition 3. If E_1 is countably generated and $[F, b]$ is compact for all $b \in B$, then there exists an odd E_1 connection for F .

If (E_1, φ_1, F_1) is a Kasparov (A, B) -module, (E_2, φ_2, F_2) a Kasparov (B, C) -module, G an F_2 connection for E_1 , then $(E_{12}, \text{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$ is a Kasparov $(\mathbb{K}_B(E_1), C)$ -module.

The existence statement stems from [CS84]; the second fact was stated in [Ska84], Proposition 9.

The composition product is given in terms of the representatives of the cycles involved: If (φ_1, E_1, F_1) is a Kasparov (A, B) -module and (φ_2, E_2, F_2) a Kasparov (B, C) -module, then a Kasparov (A, C) -module $(E_1 \otimes_B E_2, \varphi_1 \otimes 1, F_{12})$ is called a product of (E_1, φ_1, F_1) and (E_2, φ_2, F_2) if

- (i) F_{12} is an E_1 connection for F_2 (connection condition)
- (ii) For all $a \in A$, $\varphi_1(a) \otimes 1[F_1 \otimes 1, F_{12}]\varphi_1(a)^* \otimes 1$ is positive in the quotient $\mathbb{B}_C(E_{12})/\mathbb{K}_C(E_{12})$ (positivity condition).

The set of operators F_{12} satisfying the above conditions will be denoted $F_1 \sharp F_2$.

Using Kasparov's technical theorem, one can show that a product as above always exists if A is separable, is unique up to operator homotopy, and passes to homotopy classes (cf. [Ska84]).

Recall also that a Hilbert B -module E is called full if the linear span of $\langle E|E \rangle$ is dense in B .

Definition 4. Let A and B be graded C^* -algebras. A graded Morita(-Rieffel) equivalence between A and B is given by a graded full Hilbert B -module E , called the equivalence bimodule, and a graded isomorphism $\varphi : A \rightarrow \mathbb{K}_B(E)$.

We identify A with $\mathbb{K}(E)$ and drop the isomorphism φ . If E is a graded Morita equivalence bimodule from $A = \mathbb{K}_B(E)$ to B , then we define the $(B, \mathbb{K}(E))$ -module $E^* := \mathbb{K}(E, B)$. The $\mathbb{K}(E)$ -valued scalar product is simply $\langle T|S \rangle := R^*S$, and this makes E^* into a graded Hilbert $\mathbb{K}(E)$ -module.

Let A and B be separable. Then the class $[(E, \text{id}_A, 0)]$ of the equivalence bimodule yields a KK equivalence from A to B with inverse $[(E^*, \text{id}_B, 0)]$.

Conversely, any given full Hilbert B -module E may be viewed as a graded Morita equivalence from $\mathbb{K}_B(E)$ to B .

If $y = (E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, E_1 is a Hilbert B -module, and w denotes the Kasparov module defined by the Morita equivalence determined by E_1 , then the operator G in a product $w \cap x$ is exactly an E_1 -connection for F_2 , as the positivity condition is trivially satisfied.

If $x = (E_1, \varphi_1, F_1)$ and $v = (E_1^*, \text{id}_B, 0)$ is the inverse of w , then the product $x \cap v$ is represented by $(\mathbb{K}_B(E_1), \varphi_1, F_1)$, where the bounded operators on E are considered to act on the Hilbert $\mathbb{K}_B(E_1)$ -module by multiplication. This is easily seen by using the explicit form of the isomorphism $U : E_1 \hat{\otimes}_B E_1^* \rightarrow \mathbb{K}_B(E_1)$ given above, as $UTU^{-1}(|\xi\rangle\langle\eta|) = |T\xi\rangle\langle\eta|$ for all $T \in \mathbb{B}_B(E)$. Hence compact operators on E act again by compact operators on $\mathbb{K}_B(E)$, and therefore $(\mathbb{K}_B(E_1), \varphi, F_1)$ does indeed define a cycle, the connection condition is obvious, and positivity follows from $a[F_1, F_1]a^* = a(2F_1^2)a^* = aa^*$ modulo compacts.

We fix two Kasparov bimodules $(E_1, \varphi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, and denote their classes in KK by x and y . The module E_1 is seen as a Morita equivalence from $\mathbb{K}_B(E_1)$ to B , whose class in KK we denote by w , and its inverse by v . Let $(\alpha, \bar{\alpha}) : A \rightrightarrows D \supseteq \mathbb{K}_B(E_1)$ be the quasihomomorphism associated to $x' := x \cap v$, and recall that $y' := w \cap y$ may be viewed as the class of the Kasparov module defined via an E_1 connection for F_2 . If we define D_α as the sub- C^* -algebra of $A \oplus D$ generated by $(a, \alpha(a))$ and $0 \oplus B$, $a \in A$, we obtain the double split short exact sequence

$$0 \longrightarrow \mathbb{K}_B(E_1) \xrightarrow{\iota} D_\alpha \begin{array}{c} \xleftarrow{\text{id}_A \oplus \bar{\alpha}} \\ \xrightarrow{\text{id}_A \oplus \alpha} \end{array} A \longrightarrow 0$$

which in turn, by split exactness of KK , yields a long exact sequence

$$0 \longrightarrow KK(A, C) \longrightarrow KK(D_\alpha, C) \xrightarrow{\iota^*} KK(\mathbb{K}_B(E_1), C) \longrightarrow 0.$$

We may thus assume that $y' = \iota^* z$ for some $z \in KK(D_\alpha, C)$. We claim that $\alpha^*(z) - \bar{\alpha}^*(z) = y \cap x$. This follows as $KK((\alpha, \bar{\alpha}), C)$ is multiplication by x' on the left, and therefore

$$x \cap y = x' \cap y' = KK((\alpha, \bar{\alpha}), C)(y') = (\alpha^* - \bar{\alpha}^*)(\iota^*)^{-1} \iota^*(z) = (\alpha^* - \bar{\alpha}^*)(z).$$

Calculating a representative for the last expression, we have thus proved:

Theorem 5. *Let $x \in KK(A, B)$, $y = [(E_2, \varphi_2, F_2)] \in KK(B, C)$. Then the Kasparov product of x and y may be defined by*

- (i) representing x as a quasihomomorphism $(\alpha, \bar{\alpha}) : A \rightrightarrows \mathbb{B}_B(E_1) \supseteq \mathbb{K}_B(E_1)$.
- (ii) choosing an E_1 connection G for F_2
- (iii) lifting the Kasparov $(\mathbb{K}_B(E_1), C)$ -module $(E_1 \otimes_B E_2, \text{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$ along the canonical inclusion of $\mathbb{K}_B(E_1) \rightarrow D_\alpha$ to a Kasparov (D_α, C) -module $(\tilde{\varphi}, \tilde{E}, \tilde{G})$,
- (iv) and setting

$$x \cap y := \left[\left(\begin{pmatrix} \tilde{\varphi} \circ \alpha & \\ & \tilde{\varphi} \circ \bar{\alpha} \circ \varepsilon \end{pmatrix}, \tilde{E} \oplus \tilde{E}^{op}, \begin{pmatrix} \tilde{G} & \\ & -\tilde{G} \end{pmatrix} \right) \right] \in KK(A, C)$$

where E^{op} denotes the Hilbert B -module E with inversed grading, and ε the grading operator on E .

Here (iii) means exactly that the quasihomomorphism

$$Qh(E_1 \otimes_B E_2, \text{id}_{\mathbb{K}_B(E_1)} \otimes 1, G)$$

extends to a quasihomomorphism on the larger algebra D_α ; note that the class of the cycle $x \cap y$ as defined above is independent of the choice of the extension.

3 Reduction of quasihomomorphisms and a construction of the Kasparov product

For a given linear map $\varphi : A \rightarrow \mathbb{B}_B(E)$, where E is a Hilbert B -module, we define $E^\infty := \bigoplus_{n=1}^\infty E$, and $\varphi^\infty : A \rightarrow \mathbb{B}_B(E^\infty)$ as the diagonal action of φ .

Proposition 6. *The class of every quasihomomorphism is represented by a quasihomomorphism of the form $(\alpha, \text{Ad}_U \circ \alpha)$, where U is a unitary.*

Proof. Let $(\alpha, \bar{\alpha}) : A \rightrightarrows \hat{B} \supseteq B$ be a quasihomomorphism. We may assume that $\hat{B} = \mathbb{B}_B(E)$ and $B = \mathbb{K}_B(E)$ for some Hilbert B -module E . We may replace $(\alpha, \bar{\alpha})$ by

$$(\alpha \oplus \alpha^\infty \oplus \bar{\alpha}^\infty, \alpha \oplus \alpha^\infty \oplus \bar{\alpha}^\infty) : A \rightarrow \mathbb{B}_B(E \oplus E^\infty \oplus E^\infty) \supseteq \mathbb{K}_B(E \oplus E^\infty \oplus E^\infty)$$

because $(\alpha^\infty \oplus \bar{\alpha}^\infty, \alpha^\infty \oplus \bar{\alpha}^\infty)$ is degenerate.

Now let U be the unitary on $E \oplus E^\infty \oplus E^\infty$ that maps

$$(\xi_0, (\xi_1, \xi_2, \dots), (\eta_1, \eta_2, \dots)) \rightarrow (\xi_1, (\xi_2, \xi_3, \dots), (\xi_0, \eta_1, \eta_2, \dots)).$$

Then

$$(\alpha(a) \oplus \alpha^\infty(a) \oplus \bar{\alpha}^\infty(a))U = U(\bar{\alpha}(a) \oplus \alpha^\infty(a) \oplus \bar{\alpha}^\infty(a)).$$

□

Definition 7. Let A and B be C^* -algebras, $\beta : A \rightarrow \mathbb{B}(\mathcal{H}_B)$ a $*$ -homomorphism such that for every $*$ -homomorphism $\alpha : A \rightarrow \mathbb{B}(\mathcal{H}_B)$ there exists a unitary U with $\alpha \oplus \beta = U^* \beta U$ modulo compact operators. Then β will be called *absorbing*.

The following theorem was proved in [Kas80a]:

Theorem 8 (Kasparov-Voiculescu). Let A and B be separable C^* -algebras and $\beta_0 : A \rightarrow \mathbb{B}(\mathcal{H})$ a faithful representation of A such that $(\tilde{\beta}_0)^{-1}(\mathbb{K}(\mathcal{H})) = \{0\}$. We denote by β the inclusion of A into $\mathbb{B}_B(\mathcal{H}_B)$ obtained from β_0 by viewing $\mathbb{B}(\mathcal{H})$ as a subalgebra of $\mathbb{B}_B(\mathcal{H}_B)$. If either A or B is nuclear, then β is absorbing.

In general, there is a result of Thomsen from [Tho01], Theorem 2.7, which shows that for A and B separable, there is an absorbing homomorphism from A into the stable multiplier algebra $\mathcal{M}(B \otimes \mathbb{K})$ of B .

Lemma 9. Let $(\alpha, \alpha^U) : A \rightrightarrows \hat{B} \supseteq B$ be a quasihomomorphism, and $\beta : A \rightarrow \hat{B}$ a homomorphism such that $\alpha(a) - \beta(a) \in B$ for all a . Then (β, β^U) is a quasihomomorphism equivalent to (α, α^U) .

Proof. Using the usual rotation matrices, we obtain a path of unitaries

$$U_t := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} U & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Reparametrizing, we get a homotopy

$$(\alpha \oplus \beta, \text{Ad}_{U_t} \circ \alpha \oplus \beta)$$

of the quasihomomorphisms $(\alpha, \text{Ad}_U \circ \alpha) \oplus (\beta, \beta)$ and $(\alpha, \alpha) \oplus (\beta, \text{Ad}_U \circ \beta)$ \square

Proposition 10. Let $\beta : A \rightarrow \mathbb{B}(\mathcal{H}_B)$ be absorbing. Then every element of $KK(A, B)$ is represented by a quasihomomorphism of the form

$$(\beta, \text{Ad}_U \circ \beta) : A \rightarrow \mathbb{B}_B(\mathcal{H}_B) \supseteq \mathbb{K} \otimes B,$$

where $U \in \mathbb{B}_B(\mathcal{H}_B)$ is a unitary.

Proof. By Proposition 6, we may assume that we are given a quasihomomorphism $(\alpha, \alpha^U) : A \rightrightarrows \mathbb{B}_B(\mathcal{H}_B) \supseteq \mathbb{K}_B(\mathcal{H}_B)$, where U is a unitary in $\mathbb{B}_B(\mathcal{H}_B)$. Let V be a unitary such that $\alpha \oplus \beta = V^* \beta V$. Then we get

$$(\alpha, \alpha^U) \sim (\alpha \oplus \beta, \alpha^U \oplus \beta) \sim (\beta^V, \beta^{V(U \oplus 1)}) \sim (\beta, \beta^{V(U \oplus 1)V^*})$$

by the above Lemma. \square

Corollary 11. *Let A, B, C be separable C^* -algebras, β as in the above Proposition absorbing, $(\gamma, \bar{\gamma})$ a quasihomomorphism from B to C . Then it suffices to find an extension of $(\gamma, \bar{\gamma})$ to the one algebra D_β , in order to calculate explicitly all products of $(\gamma, \bar{\gamma})$ with elements from $KK(A, B)$ (as in 5).*

One can use these ideas to construct the Kasparov product in good cases:

Let $(\alpha, \bar{\alpha}) : A \rightrightarrows \mathbb{B}(B \otimes \mathcal{H}) \supseteq B \otimes \mathbb{K}(\mathcal{H})$ be a quasihomomorphism, where $\bar{\alpha} = 1 \otimes \pi$ is induced by a representation π of A on \mathcal{H} with $\pi(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$. Let $(\beta, \bar{\beta}) : B \rightrightarrows \hat{C} \supseteq C$ be another quasihomomorphism. We may extend $(\beta, \bar{\beta})$ to a quasihomomorphism

$$(\beta', \bar{\beta}') : 1 \otimes \mathbb{B}(\mathcal{H}) + B \otimes \mathbb{K}(\mathcal{H}) \rightarrow \mathcal{M}(\hat{C} \otimes \mathbb{K}(\mathcal{H})) \supseteq C \otimes \mathbb{K}(\mathcal{H})$$

by first stabilizing and then setting $\beta'(1 \otimes T + x) := 1 \otimes T + \beta \otimes \text{id}_{\mathbb{K}}(x)$. Because $D_{\bar{\alpha}} \subseteq 1 \otimes \mathbb{B}(\mathcal{H}) + B \otimes \mathbb{K}(\mathcal{H})$, we have constructed a product. Note further that because $(\beta', \bar{\beta}')$ represents zero on the image of $\bar{\alpha}$, the product has a very simple form:

$$[\alpha, \bar{\alpha}] [\beta', \bar{\beta}'] = [\beta' \circ \alpha, \bar{\beta}' \circ \alpha].$$

In particular, if we have any two quasihomomorphisms $(\alpha, \bar{\alpha})$ from A to B and $(\beta, \bar{\beta})$ from B to C and either A or B is nuclear, then by Proposition 10 we may assume that $\bar{\alpha}$ is obtained from a faithful representation A whose image is disjoint from the compacts, and then apply the construction as above. More generally, one may construct on this way the Kasparov product for the functor KK_{nuc} from [Ska88].

This construction of the product coincides with the one by Kasparov by the preceding section.

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